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Global stability of condensation in the continuum limit for Fröhlich's pumped phonon system

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Abstract. For a continuum version of Fröhlich's pumped phonon system we show that (i) there exists a critical value of the pumping above which the stationary state displays condensation; (ii) the stationary state is *globally* stable WRT perturbations; and (iii) the relaxation times for the condendsate diverge at the onset of condensation.

1. Introduction

In 1968 Fröhlich proposed a model of coherent excitations in biological systems [1]. The model comprises a finite number of polarisation waves immersed in a heat bath, but maintained away from equilibrium by external pumping. Fröhlich described this by means of a non-linear kinetic equation for the occupation numbers (equation (1.1) below) and argued that for sufficiently strong pumping the stationary state undergoes Bose condensation into the mode of lowest frequency.

In this paper we examine a continuum analogue of Fröhlich's equation, i.e. we consider infinitely many modes. Our principal results are that (i) there is a critical value of the pumping, above which the stationary distribution displays condensation (in the sense that it has a Dirac measure at the lowest energy); (ii) the stationary distribution is *globally* stable with respect to pertubations (so that it is the terminal state of the evolution for all initial conditions). This result is achieved by construction of a Lyapounov functional for the evolution. Finally, (iii) for the linearised evolution about the stationary state, relaxation times diverge as the pumping approaches its critical value from below.

1.1. Fröhlich's model

We briefly review the original formulation of the model. Let there be V modes with frequencies ω_k: $0<\omega_1\leq\omega_2\leq\ldots\leq\omega_V$. Denote by n_k the occupation number of the kth mode. The system heat-bath interaction is assumed to lead to spontaneous emission and absorbtion of phonons (with transition probabilities ξ_k) and two-phonon exchanges (with probability χ_{jk}/V). Detailed balance at heat-bath temperature T is assumed for these processes. Energy is pumped into the kth mode at a rate s_k. In units for which $\hbar=k_BT$ these assumptions give rise to the following kinetic equation for the n_k:

$$\mathrm{d}n_k/\mathrm{d}t=s_k-\xi_k[n_k\,\mathrm{e}^{\omega_k}-(1+n_k)]-\sum_{j=1}^{V}\chi_{jk}[n_k(1+n_j)\,\mathrm{e}^{\omega_k}-n_j(1+n_k)\,\mathrm{e}^{\omega_j}]. \qquad (1.1)$$

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Assuming the transition probabilities to be uniform (i.e. $s_k = s$, $\xi_k = \xi$ and $\chi_{jk} = \chi$), Fröhlich found a self-consistent expression for the stationary distribution of (1.1) which we denote by m_k:

$$m_k = [1 + (\xi/\chi)(1 - e^{-\mu})] \frac{1}{\exp(\omega_k - \mu) - 1} \tag{1.2}$$

where μ is the 'effective chemical potential', determined from (1.1) by the requirement that $\Sigma_k \dot{m}_k = 0$:

$$\xi + s = \frac{\xi}{V} \sum_{k=1}^{V} m_k (e^{\omega_k} - 1). \tag{1.3}$$

(That such a unique value of μ exists follows from the fact that for all k the function $\mu \mapsto m_k$ defined by (1.2) is increasing.) Define the density $\rho = V^{-1} \Sigma_k m_k$. From (1.3) we have that $K_1 \rho \leq \xi + s \leq K_2 \rho$ for some $0 < K_1 < K_2 < \infty$, so that ρ must increase indefinitely with s. From (1.2), this is only possible if μ approaches ω_1 from below.

This is the essence of Fröhlich's model. Loosely speaking, as the pumping s increases, most of the supplied energy is channelled into the lowest energy mode, while all other modes become saturated. This behaviour is a true non-equilibrium effect (if $s = 0$, the stationary distribution is Planckian), which depends on the collective behaviour of the phonon modes (if $\chi = 0$ all the modes are occupied to roughly the same degree).

1.2. The continuum model

The experience from equilibrium statistical mechanics is that phase transitions become clear-cut only at infinite volume [2]. Thus we seek to generalise (1.1) to treat a continuum of mode frequencies. Specifically, let X be a closed and bounded interval of the positive real line. X is the set of frequencies of the continuum of modes. The sum over finitely many modes in (1.1) is replaced by integration against a measure $\nu(dx)$. $d\nu/dx$ is to be understood as a density of states. (We will relate ν to the dispersion relation for a class of Debye type models.)

We propose the following analogue of (1.1). The vector of occupation numbers $n_k(t)$ goes over to a one-parameter family of measures $\{\varphi_t\}_{t \in R}$ on X. Let f be an arbitrary continuous function on X. Then we have

$$\begin{aligned} \frac{d}{dt} \int_X \varphi_t(dx) f(x) &= \int_X \nu(dx) s(x) f(x) - \int_X [\varphi_t(dx)\, e^x \xi(x) f(x) - (\varphi_t(dx) + \nu(dx)) \xi(x) f(x)] \\ &\quad - \int_X \int_X \varphi_t(dx)(\varphi_t(dy) + \nu(dy))[e^x \chi(x, y)(f(x) - f(y))] \end{aligned} \tag{1.4}$$

where $s(x)$, $\xi(x)$ and $\chi(x, y)$ are the continuum analogues of the discrete coupling functions. Thus differentiability of φ_t is understood in the weak-* sense.

Remark. Elsewhere [3], we show how (1.4) can be derived from a family of quantum dynamical semigroups [4] (one for each volume V) associated with (1.1). The corresponding hierarchy of reduced density matrices has the property that it decorrelates in the thermodynamic limit: each order of the hierarchy can be written as a product of solutions of (1.4).

1.3. Condensation

Henceforth we will confine our attention to a subset of all possible values of the coupling functions s, ξ and χ (see (2.2)). In § 2 we will see that for appropriate measures ν the mean pumping s has a critical value s_c above which the stationary solution of (1.4) has a Dirac measure at the lowest energy in X. Thus we can speak of generalised non-equilibrium Bose condensation, and the condensate density $(=\max\{0, (s-s_c)\})$ acts as an order parameter for the order-disorder transition. The most important property of the infinite volume model is that a finite fraction of the total density is found at the lowest energy whenever $s > s_c$. In the finite models we can only say that since n_1 can be made as large as desired.

Now, condensation as applied to Bose gases is usually understood from equilibrium statistical mechanics. For example, one constructs the grand canonical ensemble for the free gas confined to an increasing family of boxes and demonstrates the existence of a critical density [5]. However, the free Bose gas has also been treated dynamically: in [6] a non-linear kinetic equation for the gas coupled to a heat bath is found. The stationary state has a Dirac measure at zero energy. Thus condensation is not a solely equilibrium notion.

1.4. Dynamics and global stability

In § 3 we examine the dynamics generated by (1.4) in the space of measures on X and settle some technical questions concerning the existence of a bona fide solution. In § 4 we proceed to the main result of the paper, namely that the stationary state is *globally stable* with respect to perturbations. This is achieved by construction of a Lyapounov function for the evolution, i.e. a continuous function bounded below which decreases along trajectories. The main task here is to find a suitable monotonic function. To prove continuity, we adapt some results from [6-9].

We mention here two previous pieces of work on stability. Firstly, in [10], *local* stability of Fröhlich's original equation (1.1) was investigated via a series expansion in s^{-1}. To leading order only the eigenvalues of the linearised evolution about the stationary state were found to be negative. Subsequently, in [11], the present author proved *global* stability for a restricted class of equations of the form (1.1) by construction of a Lyapounov function. The present paper may be seen as an extension of the results in [11] to the measure theoretic case.

1.5. Linearised evolution

Finally, in § 5, we investigate the spectrum of the linearised evolution about the stationary state. Our result is that relaxation times diverge as s approaches s_c from below. This is to be expected in the light of Haken's general framework for order parameters far from equilibrium [12]. The divergence of relaxation times corresponds to the fact that the disordered phase (i.e. that with zero stationary condensate density) becomes unstable as we approach the phase transition. This behaviour does not appear to be accessible from existing calculations on finite-volume models.

2. The stationary distribution: condensation

2.1. The models

We consider a class of Debye models with dispersion. We work in d dimensions. Let $K=[0, \bar{k}]$: $\bar{k}<\infty$ contain the norms of the momenta of all modes under consideration. Let $X=[\underline{x}, \bar{x}]$: $0<\underline{x}<\bar{x}<\infty$ contain all the energies. Then we assume a function $\omega: K \rightarrow X$ (the dispersion relation) with the following properties:

(a) $\omega(K)=X$;

(b) ω is strictly monotonic increasing;

(c) in some open interval N of K containing 0, then $\omega(k)=\omega(0)+k^{\kappa}\eta(k)$ with $\kappa>1$ and η continuously differentiable on N with $\eta(0)>0$.

Note that (a) and (b) together imply that $\omega(0)=\underline{x}$ and $\omega(\bar{k})=\bar{x}$. We are now able to define the measure ν on X, against which integration replaces the sum of the finite model: let $f \in C(X)$. Then define the linear functional ν on $C(X)$ by

$$\nu(f)=\frac{d}{\bar{k}^{d}} \int_{0}^{\bar{k}} \mathrm{d}k\, k^{d-1}(f \circ \omega)(k). \tag{2.1}$$

Note that ν is normalised so that $\nu(\mathbf{1})=1$ (define $\mathbf{1}(x)=1$ on X). Clearly ν is positive and hence continuous, so by the Riesz representation theorem [13] it defines a Baire measure (with a unique regular Borel extension which we also denote by ν) on the compact set X.

2.2. Existence of condensation

Hereafter we restrict ourselves to the following choice of coupling functions s, ξ and χ:

$$\chi(x, y)=\gamma \exp[-(x+y)] \qquad \xi(x)=(\mathrm{e}^{x}-1)^{-1} \qquad s(x)=s-\xi(x) \tag{2.2}$$

where $\gamma>0$ and $s>(\mathrm{e}^{\underline{x}}-1)^{-1}$. γ and s are now the parameters of the theory. For clarity we write out (1.4) with the parameters (2.2):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{X} & \varphi_{t}(\mathrm{d}x) f(x) \\
& =\int_{X}(Q \varphi_{t})(\mathrm{d}x) f(x) \\
& := s \int_{X} \nu(\mathrm{d}x) f(x)-\int_{X} \varphi_{t}(\mathrm{d}x) f(x) \\
& \quad -\gamma \int_{X} \int_{X} \varphi_{t}(\mathrm{d}x)(\varphi_{t}(\mathrm{d}y)+\nu(\mathrm{d}y)) e(y)(f(x)-f(y))
\end{aligned} \tag{2.3}$$

where $e(x)$ is the function $\exp(-x)$ on X. Then we have the following form for the stationary measure of (2.3).

Proposition 1. Let $\mu \in(-\infty, \underline{x}]$, and let m_{μ} denote the function

$$m_{\mu}(x)=\frac{1+\gamma e(x)}{\gamma e(\mu)-\gamma e(x)} \tag{2.4}$$

on X. Then for dimension $d > \kappa$, the critical pumping $s_c = \nu(m_{\underline{x}})$ is finite and the stationary measure for (2.3) at pumping s is

$$\varphi^s(\ ) = (s - s_c)^+ \delta_{\underline{x}}(\ ) + \nu(m_\mu \cdot) \tag{2.5}$$

where $(p)^+ = \frac{1}{2}(p + |p|)$, $\delta_{\underline{x}}$ is the Dirac measure at $\underline{x}$ and μ is the unique solution of

$$s - (s - s_c)^+ = \nu(m_\mu). \tag{2.6}$$

Proof. First we consider the finiteness of s_c. For all $\mu \in (-\infty, \underline{x}]$, m_μ is bounded and continuous on $[\underline{x} + \varepsilon, \bar{x}]$ for any $\varepsilon > 0$, and hence integrable against ν there. So to demonstrate that s_c is finite, we need only show that $m_{\underline{x}}(x)$ is integrable in some neighbourhood of X containing $\underline{x}$. By assumption (c) above, we can take the Taylor expansion of ω in the neighbourhood N:

$$\omega(k) - \omega(0) = k^\kappa(\eta(0) + k\eta'(\hat{k})) \tag{2.7}$$

for $k \in N$ and some $\hat{k} \in [0, k]$. Now pick some non-zero λ in N. Then for all $k \in [0, \lambda]$

$$0 \leqslant \omega(k) - \omega(0) \leqslant C_\lambda k^\kappa \tag{2.8a}$$

where

$$C_\lambda = \eta(0) + \lambda \max\left\{0, \sup_{k \in [0,\lambda]} \eta'(k)\right\}. \tag{2.8b}$$

Combining (2.8) and (2.1) we see that

$$\int_{\underline{x}}^{\omega(\lambda)} \nu(\mathrm{d}x) m_{\underline{x}}(x) \leqslant \int_0^\lambda \mathrm{d}k\, k^{d-1} \frac{1+\gamma}{\gamma e(\underline{x})(1 - e(C_\lambda k^\kappa))}. \tag{2.9}$$

The RHS of (2.9) is integrable whenever $d > \kappa$. A similar estimate shows that the converse is true: the LHS is never integrable when $d \leqslant \kappa$.

Next consider the form of the stationary measure. For $d > \kappa$, the statement that (2.6) has a unique solution μ for $s < s_c$ follows from the fact that $m_\mu(x)$ is a strictly increasing function of μ, continuous uniformly for x: hence the function $\mu \mapsto \nu(m_\mu)$ is also continuous and strictly increasing.

All that remains is to verify the form (2.5). We rewrite (2.3) as

$$(Q\varphi^s)(f) = \gamma(\varphi^s \otimes (\varphi^s + \nu))((1 \otimes ef) - (f \otimes e)) + s\nu(f) - \varphi^s(f) \tag{2.10}$$

for all f in $C(X)$. Inverting (2.4) we find that

$$\gamma e(x) = (1 + \gamma e(\mu)) M_\mu(x) - 1 \qquad \text{with } M_\mu(x) = m_\mu(x)/(1 + m_\mu(x)) \tag{2.11}$$

and so

$$(Q\varphi^s)(f) = (1 + \gamma e(\mu))[\varphi^s(1)(\varphi^s + \nu)(M_\mu f) - (\varphi^s + \nu)(M_\mu)\varphi^s(f)] + (s - \varphi^s(1))\nu(f). \tag{2.12}$$

Now by (2.5) we see that

$$(\varphi^s + \nu)(M_\mu \cdot) = (s - s_c)^+ \delta_{\underline{x}}(M_\mu \cdot) + \nu(m_\mu \cdot) \tag{2.13}$$

and so

$$\begin{aligned}(Q\varphi^s)(f) = (1 + \gamma e(\mu))[&((s - s_c)^+ + \nu(m_\mu))((s - s_c)^+ M_\mu(\underline{x}) f(\underline{x}) + \nu(m_\mu f)) \\ &- ((s - s_c)^+ f(\underline{x}) + \nu(m_\mu f))((s - s_c)^+ M_\mu(\underline{x}) + \nu(m_\mu))] \\ &+ (s - \varphi^s(1))\nu(f).\end{aligned} \tag{2.14}$$

Equations (2.5) and (2.6) together imply that $\varphi^s(1)=s$. By multiplying out (2.14) one finds that the only possible non-zero term in $(Q\varphi^s)(f)$ is

$$(1+\gamma e(\mu))(s-s_c)^+(1-M_\mu(\underline{x}))(\nu(m_\mu f)-\nu(m_\mu)f(\underline{x})). \tag{2.15}$$

But, since when s is finite

$$(s-s_c)^+\geqslant 0 \Leftrightarrow \mu=\underline{x} \Leftrightarrow \lim_{x\to\underline{x}} M_\mu(x)=1 \tag{2.16}$$

we have that $(s-s_c)^+(1-M_\mu(\underline{x}))=0$ for all s, and hence that $(Q\varphi^s)(f)=0$.

Remarks. (1) $\varphi(1)=s$: the pumping fixes the stationary density. (2) If $\gamma=0$ then the stationary measure is $\varphi^s(f)=s\nu(f)$ for all s, so condensation will never occur.

3. Dynamics

We look for time-dependent solutions of (2.3). Bearing in mind the form of the stationary measure (2.5), we will restrict our attention to measures of the form

$$\varphi_t(\cdot)=\nu(n_t\cdot)+\sigma_t\delta_{\underline{x}}(\cdot) \tag{3.1}$$

where $n_t\in L^1(X,\mathrm{d}\nu)^+$ and $\sigma_t\in\boldsymbol{R}^+$, so that the time-dependent measure is the sum of a condensate term, and a measure absolutely continuous with respect to ν.

We consider the Banach space $\mathscr{B}=L^1(X,\mathrm{d}\nu)\oplus\boldsymbol{R}$ with norm $\|n\oplus\sigma\|=\|n\|_{L^1}+|\sigma|$. Inserting (3.1) into (2.3) yields the system

$$\dot n_t(x)=s-n_t(x)+\gamma\sigma_t[(e(x)-e(\underline{x}))n_t(x)+e(x)]$$
$$+\gamma\int_X \nu(\mathrm{d}y)[n_t(y)(1+n_t(x))e(x)-n_t(x)(1+n_t(y))e(y)] \tag{3.2a}$$

$$\dot\sigma_t=-\sigma_t\int_X \nu(\mathrm{d}x)\{1+\gamma[(e(x)-e(\underline{x}))n_t(x)+e(x)]\}. \tag{3.2b}$$

We must show that (3.2) generates a bona fide global solution in $\mathscr{B}^+$.

Lemma 2. For each initial condition $n_0\oplus\sigma_0$ in $\mathscr{B}$, (3.2) generates a local solution $\{n_t\oplus\sigma_t\}_{0\leqslant t\leqslant\tau}$ where τ depends only on $\|n_0\oplus\sigma_0\|$.

Proof. This follows from the fact that in any norm ball B of $\mathscr{B}$ the derivative (3.2) is bounded and uniformly continuous WRT the $\|\cdot\|$ topology on $\mathscr{B}$. The integral equation associated with (3.2) yields a contraction mapping in $\mathscr{C}([0,\tau];B)$ for sufficiently small τ in the usual manner. The unique fixed point of this map is the local solution [14].

We note that for this local solution, the density $\rho_t=\sigma_t+\int_X \nu(\mathrm{d}x)n_t(x)$ obeys the closed equation

$$\dot\rho_t=s-\rho_t \tag{3.3}$$

and hence relaxes to its stationary value s. In particular, we have the bound $\rho_t\leqslant\max\{s,\rho_0\}$.

The local value of σ_t has the following property.

Lemma 3.

$$\sigma_0 = 0 \Rightarrow \sigma_t = 0 \qquad t \in [0, \tau] \tag{3.4a}$$

$$\sigma_0 > 0 \Rightarrow \sigma_t > 0 \qquad t \in [0, \tau]. \tag{3.4b}$$

Proof. From (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_t^2 = -2\sigma_t^2 \int_X \nu(\mathrm{d}x)\{1+\gamma[(e(x)-e(\underline{x}))n_t(x)+e(x)]\} \tag{3.5}$$

so since $\|n_t \oplus \sigma_t\|$ is bounded in $[0, \tau]$

$$-\sigma_t^2 K_2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t^2 \leq \sigma_t^2 K_1 \tag{3.6}$$

for some positive K_1 and K_2. Hence, by Gronwall's lemma [14]

$$\sigma_0^2 \exp(-K_2 t) \leq \sigma_t^2 \leq \sigma_0^2 \exp(K_1 t) \tag{3.7}$$

whence (3.4).

Lemma 4. The local solution preserves the positivity of initial conditions.

Proof. We split up the derivative (3.2a) into a linear and a non-linear part: $\dot{n}_t = \mathrm{L}n_t + \mathrm{N}n_t$ where

$$(\mathrm{L}n_t)(x) = -(1+\gamma ac)n_t(x) - \gamma \int_X \nu(\mathrm{d}y)[n_t(x)e(y) - n_t(y)e(x)] \tag{3.8a}$$

$$(\mathrm{N}n_t)(x) = s + \gamma c(a-\rho_t)n_t(x) + \gamma\sigma_t[(c+e(x)-e(\underline{x}))n_t(x)+e(x)]$$
$$+\gamma \int_X \nu(\mathrm{d}y)n_t(x)n_t(y)(c+e(x)-e(y)) \tag{3.8b}$$

where $a = 2\max\{\rho_0, s\}$ and c is any real number such that $c+e(x)-e(y)>0$ for all $x, y \in X$, and ρ_t is the solution of (3.3) with initial density ρ_0. Thus $\{n_t\}_{t\in[0,\tau]}$ is the solution of the integral equation

$$n_t = (\Theta n)_t := \exp(\mathrm{L}t)n_0 + \int_0^t \mathrm{d}r \exp[\mathrm{L}(t-r)]\mathrm{N}n_r. \tag{3.9}$$

The point now is that $\exp(\mathrm{L}t)$ and N both preserve order in $\mathrm{L}^1(X, \mathrm{d}\nu)$. (This is obvious for N; we give an explicit form for $\exp(\mathrm{L}t)$ in (3.14) below.) Thus the iteration sequence

$$n_t^0(x) = 0 \qquad n_t^{i+1} = \Theta n_t^i \qquad i \in \mathrm{N} \tag{3.10}$$

is for each $t \in [0, \tau]$ an increasing sequence in $\mathrm{L}^1(X, \mathrm{d}\nu)^+$. If this sequence is norm bounded (uniformly in $[0, \tau]$) then the monotone convergence theorem implies the existence of a limit. Furthermore, since Θ is continuous, this limit will be the local solution previously constructed in lemma 2. But this is easy to arrange. Define

$$P_\tau^{(i)} := \sup_{t\in[0,\tau]}\left(\sigma_t + \int_X \nu(\mathrm{d}x)n_t^i(x)\right). \tag{3.11}$$

Then a simple estimate of (3.8) and (3.9) yields

$$P_\tau^{(i+1)} \leq \rho_0 + (u + vP_\tau^{(i)} + w(P_\tau^{(i)})^2)\frac{1-\exp[-(1+\gamma ac)\tau]}{1+\gamma ac} \tag{3.12}$$

where u, v and w are positive constants. Thus we can always choose τ small enough that

$$P_\tau^{(i)} < a \Rightarrow P_\tau^{(i+1)} < a. \tag{3.13}$$

Since the n_t^i are positive, a serves as a bound for $\|n_t^i\|$ as required.

Proposition 5. There exists a global positive solution to (2.3) for positive initial conditions.

Proof. The constants a and c can be chosen globally for a given initial condition, and so we can construct a global solution by joining together local positive solutions constructed on the intervals $[0, \tau]$, $[\tau, 2\tau]$, etc, via lemmas 2, 3 and 4.

Now we give the explicit form of $\exp(\mathrm{L}t)$:

$$(\exp(\mathrm{L}t)n)(x) = \exp[-(1+\gamma ac)t]$$

$$\times\left(n(x)\exp(-\gamma t\nu(e)) + \frac{\nu(n)}{\nu(e)}e(x)[1-\exp(-\gamma t\nu(e))]\right). \tag{3.14}$$

This is easily verified by differentiation. $\exp(\mathrm{L}t)$ is positive and linear, hence order preserving. We will need this explicit form in the next section.

4. Global stability

In this section we prove that the stationary solution (2.5) is the terminal state for every initial condition of the form (3.1).

4.1. *Lyapounov function: monotonicity*

First, we recall from (3.3) that ρ_t relaxes to its stationary value s. Thus $\mathscr{B}^+$ divides into three regions invariant under the evolution

$$K^+ = \{n \oplus \sigma \in \mathscr{B}^+ : \rho \geqslant s\} \tag{4.1a}$$

$$K^0 = \{n \oplus \sigma \in \mathscr{B}^+ : \rho = s\} \tag{4.1b}$$

$$K^- = \{n \oplus \sigma \in \mathscr{B}^+ : \rho \leqslant s\}. \tag{4.1c}$$

Note that $\mathscr{B}^+ = K^+ \cup K^0 \cup K^-$. In what follows we work at given pumping s with associated chemical potential μ. We now define *on the whole of* $\mathscr{B}^+$ the functional

$$\Lambda^0(n\oplus\sigma) = -\sigma\log\left(\frac{m_\mu(\underline{x})}{1+m_\mu(\underline{x})}\right) + \int_X \nu(\mathrm{d}x)F(n(x);\, m_\mu(x)) \tag{4.2}$$

where $F: \boldsymbol{R}^+ \times \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is given by

$$F(p;q) = p\log(p/q) - (1+p)\log\left(\frac{1+p}{1+q}\right). \tag{4.3}$$

We shall see that Λ^0 is a Lyapounov functional for the restriction of the evolution to the invariant set K^0. We shall have to build separate extensions Λ^+ and Λ^- for the evolution on K^+ and K^-.

Proposition 6. The Lyapounov functional on K^0. Let $\hat{\sigma}=(s-s_c)^+$, i.e. the stationary condensate density. Then we have the following.

(a) For all $n\oplus\sigma\in\mathscr{B}^+\supset K^0$

$$\Lambda^0(n\oplus\sigma)\geqslant\Lambda^0(m_\mu\oplus\hat{\sigma})=0 \tag{4.4}$$

with equality iff $n\oplus\sigma=m_\mu\oplus\hat{\sigma}$.

(b) For any initial condition $n_0\oplus\sigma_0$ in K^0

$$\Lambda^0(n_t\oplus\sigma_t)-\Lambda^0(n_\delta\oplus\sigma_\delta)=-\int_\delta^t \mathrm{d}\tau\,\Gamma^0(n_\tau\oplus\sigma_\tau) \tag{4.5}$$

for any $\delta\in(0,t]$, the functional Γ^0 defined as

$$\Gamma^0(n_t\oplus\sigma_t)=\sigma_t(1+\gamma e(\mu))\int_X \nu(\mathrm{d}x)n_t(x)[R_t(x)-M_\mu(\underset{\sim}{x})]\log\left(\frac{R_t(x)}{M_\mu(\underset{\sim}{x})}\right)$$
$$+\tfrac{1}{2}(1+\gamma e(\mu))\int_X\int_X \nu(\mathrm{d}x)\nu(\mathrm{d}y)n_t(x)n_t(y)[R_t(y)-R_t(x)]\log\left(\frac{R_t(y)}{R_t(x)}\right) \tag{4.6}$$

where

$$R_t(x)=\frac{M_\mu(x)}{N_t(x)}\qquad N_t(x)=\frac{n_t(x)}{1+n_t(x)}\qquad M_\mu(x)=\frac{m_\mu(x)}{1+m_\mu(x)}. \tag{4.7}$$

(c) $\qquad\Gamma^0(n\oplus\sigma)\geqslant 0\qquad \forall n\oplus\sigma\in\Lambda^0.$

Proof of (a). This follows from the observations that

(i) $F(p;q)$ is convex in p, and for a given q takes its minimum value, zero, at $p=q$;

(ii) $F(p;q)\leqslant p/q$ so that the integrand in (4.3) is integrable whenever n is; and

(iii) $-\sigma\log M_\mu(\underset{\sim}{x})\geqslant 0$ and $\hat{\sigma}\log M_\mu(\underset{\sim}{x})=0$ since $\hat{\sigma}\geqslant 0\Leftrightarrow\mu=\underset{\sim}{x}\Leftrightarrow M_\mu(\underset{\sim}{x})=1$.

Proof of (b). First, we must show that $\Lambda^0(n_t\oplus\sigma_t)$ is differentiable. This is problematic since $F(p;q)$ is not differentiable WRT p at $p=0$. However, we can circumvent this case by means of the following lemma, which can be used to show that the derivative of Λ^0 exists after an arbitrarily short length of time.

Lemma 7. For all initial conditions $n_0\oplus\sigma_0$, and all $\delta>0$, $n_t(x)$ is almost everywhere $[\nu]$ bounded away from zero uniformly for all $t\geqslant\delta$.

Proof. Recall that the iteration sequence (3.10) increases monotonically to its limit, so that

$$n_t(x)\geqslant n_t^1(x)=(\exp(\mathrm{L}t)n_0)(x)$$
$$+\int_0^t \mathrm{d}r(\exp(\mathrm{L}r)Nn_{t-r}^0)(x)\qquad\text{almost everywhere }[\nu]. \tag{4.8}$$

But $n_t^0=0$ for all x, so that

$$N(n_t^0)(x)=s+\gamma\sigma_t e(x). \tag{4.9}$$

Using the explicit form (3.14) for exp(Lt) and retaining only the first term,

$$(\exp(\mathrm{L}t)\mathbf{1})(x) \geq \exp(-bt) \qquad b = 1+\gamma(ac+\nu(e)). \tag{4.10}$$

Combining (4.8)-(4.10) and performing the integration over r

$$n_t(x) \geq s[1-\exp(-bt)]/b \qquad \text{almost everywhere } [\nu] \tag{4.11}$$

and hence the almost everywhere lower bound on $\{n_t(x)\}_{t\geq\delta}$ is $k_\delta = s(1-\exp(-b\delta))/b$.

Remark. It follows from (4.11) that

$$\log M_\mu(\bar{x}) \leq \log R_t(x) \leq 1/k_\delta + \log M_\mu(\underset{\sim}{x}) \qquad t \geq \delta. \tag{4.12}$$

Proof of (*b*) (*continued*). Differentiability of the condensate term in (4.2) is trivial. Now, $F(p; q)$ is convex in p and $\partial F(p, q)/\partial p = \log[p(1+q)/q(1+p)]$, so that for $t \geq \delta > 0$ and $h > 0$

$$\begin{aligned}(n_{t+h}(x) - n_t(x)) \log R_{t+h}(x) &\\ &\leq F(n_t(x); m_\mu(x)) - F(n_{t+h}(x); m_\mu(x))\\ &\leq (n_{t+h}(x)-n_t(x)) \log R_t(x).\end{aligned} \tag{4.13}$$

If we divide through by h, then pointwise, both sides of the inequality converge to $\dot{n}_t(x) \log R_t(x)$ as $h \to 0$. Since n_t is L^1 differentiable and $\log R_t(x)$ is almost everywhere bounded (equation (4.12)), we can use the dominated convergence theorem to achieve the same convergence under integration over X WRT ν. The same result holds with h negative, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X \nu(\mathrm{d}x) F(n_t(x); m_\mu(x)) = -\int_X \nu(\mathrm{d}x) \log R_t(x) \frac{\mathrm{d}}{\mathrm{d}t} n_t(x). \tag{4.14}$$

Expressing $m_\mu(x)$ in terms of $e(x)$, the derivatives (3.2) are written

$$\begin{aligned}\dot{n}_t(x) &= (s-\rho_t) + \sigma_t(1+\gamma e(\mu)) n_t(x)[R_t(x) - M_\mu(\underset{\sim}{x})]\\ &\quad + (1+\gamma e(\mu)) \int_X \nu(\mathrm{d}y) n_t(y) n_t(x)(R_t(x) - R_t(y))\\ \dot{\sigma}_t &= -\sigma_t(1+\gamma e(\mu)) \int_X \nu(\mathrm{d}y) n_t(y)[R_t(y) - M_\mu(\underset{\sim}{x})]\end{aligned} \tag{4.15}$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^0(n_t \oplus \sigma_t) &= (\rho_t - s)\int_X \nu(\mathrm{d}x) \log R_t(x)\\ &\quad + \sigma_t(1+\gamma e(\mu)) \int_X \nu(\mathrm{d}x) n_t(x)(R_t(x) - M_\mu(\underset{\sim}{x})) \log\left(\frac{R_t(x)}{M_\mu(\underset{\sim}{x})}\right)\\ &\quad + (1+\gamma e(\mu)) \int_X \nu(\mathrm{d}x) \int_X \nu(\mathrm{d}y) n_t(y) n_t(x)(R_t(x) - R_t(y)) \log R_t(x).\end{aligned} \tag{4.16}$$

Since $\rho_t = s$ on K^0, the first term in (4.16) vanishes, and the bounds of lemma 7 justify an exchange of integrals in the last term, yielding (4.6). Finally the integral form (4.5) follows because, since n_t and σ_t are C^1 functions of t and $R_t(x)$ is bounded, $\Gamma(n_t \oplus \sigma_t)$ is a continuous function of t.

Proof of (c). By inspection of (4.6), Γ^0 is positive.

We cannot directly extend proposition 6 to cover the whole of $\mathscr{B}^+$ since the first term in (4.16) is of indefinite sign when $\rho \neq s$. However, from the following lemma we can infer that it converges to zero.

Lemma 8. Let $\Xi(n_t) = -\int_X \nu(\mathrm{d}x) \log N_t(x)$. Then for all $t>0$, $\Xi(n_t)$ is differentiable WRT time, and

$$\dot{\Xi}(n_t) \leqslant (a+b\rho_t)-(s+c\rho_t)\Xi(n_t) \tag{4.17}$$

where $a = 1+s$, $b = \gamma e(\underline{x})$ and $c = \gamma e(\bar{x})$ and hence $\Xi(n_t)$ is unformly bounded for all $t \geqslant \delta > 0$.

Proof. The function $p \mapsto \log[(1+p)/p]$ is convex, so by lemma 7, and using an argument similar to that in proposition 6, the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X \nu(\mathrm{d}x)\log\left(\frac{1+n_t(x)}{n_t(x)}\right) = -\int_X \nu(\mathrm{d}x)\left(\frac{\dot{n}_t(x)}{n_t(x)(1+n_t(x))}\right)$$

exists for all $t>0$. Using the form (3.2a) for $\dot{n}_t(x)$:

$$\begin{aligned}\dot{\Xi}(n_t) = s\int_X \nu(\mathrm{d}x)&\left(\frac{1}{1+n_t(x)}-\frac{1}{n_t(x)}\right)\\ &+\int_X \nu(\mathrm{d}x)\frac{1}{1+n_t(x)}\left(1+\gamma\sigma_t e(\underline{x})+\gamma\int_X \nu(\mathrm{d}y)(1+n_t(y))e(y)\right)\\ &-\int_X \nu(\mathrm{d}x)\frac{1}{n_t(x)}\gamma e(x)\left(\sigma_t+\int_X \nu(\mathrm{d}y)n_t(y)\right)\end{aligned} \tag{4.18a}$$

and so since $\int_X \nu(\mathrm{d}x) = 1$,

$$\begin{aligned}\dot{\Xi}(n_t) = \int_X \nu(\mathrm{d}x)\int_X \nu(\mathrm{d}y)&\left(\frac{1+s+\gamma[e(y)(1+n_t(y))+\sigma_t e(\underline{x})]}{1+n_t(x)}\right.\\ &\left.-\frac{s+\gamma e(x)(\sigma_t+n_t(y))}{n_t(x)}\right).\end{aligned} \tag{4.18b}$$

Applying the inequality $(1+n_t(x))^{-1} \leqslant 1$ to the first term of (4.18b) and the logarithmic inequality $-1/n_t(x) \leqslant \log N_t(x)$ to the second yields (4.17). Since ρ_t is uniformly bounded and $\Xi(n_t)$ is positive, it is a simple matter to use Gronwall's lemma to show that

$$\Xi(n_t) \leqslant \Xi(n_\delta)\exp[-B(t-\delta)] + C\{1-\exp[-B(t-\delta)]\} \tag{4.19}$$

for $t \geqslant \delta > 0$, B, C finite positive constants depending on the initial condition.

Proposition 6A. The Lyapounov functional on K^+. Define on K^+ the functional

$$\Lambda^+(n\oplus\sigma) = \Lambda^0(n\oplus\sigma) + a(\rho-s) + \tfrac{1}{2}b(\rho^2-s^2) + (\rho-s)\Xi(n). \tag{4.20}$$

Then

(a) for all $n\oplus\sigma \in K^+$, $\Lambda^+(n\oplus\sigma) \geqslant \Lambda^+(m_\mu\oplus\hat{\sigma}) = 0$ with equality iff $n\oplus\sigma = m_\mu\oplus\hat{\sigma}$;
(b) for any initial condition in K^+

$$\Lambda^+(n_t\oplus\sigma_t) - \Lambda^+(n_\delta\oplus\sigma_\delta) = -\int_\sigma^t \mathrm{d}\tau\, \Gamma^+(n_\tau\oplus\sigma_\tau)$$

for any $\delta \in (0, t]$, the functional Γ^+ defined as

$$\Gamma^+(n_t \oplus \sigma_t) = \Gamma^0(n_t \oplus \sigma_t) + (\rho_t - s)\left((a + b\rho_t) - \dot{\Xi}(n_t) - \int_X \nu(\mathrm{d}x) \log M_\mu(x)\right) \qquad (4.21)$$

(c) $\Gamma^+(n_t \oplus \sigma_t) \geqslant 0$.

Proof of (a). This follows from the positivity of Λ^0(proposition 6), of Ξ and of $\rho - s$.

Proof of (b). By proposition 6 and lemma 7, Λ^+ is differentiable for all $t > 0$. Differentiating (4.20) and applying (3.3) and (4.16) (including the first term of the latter) the stated form results.

Proof of (c). Substituting (4.17) into (4.21):

$$\Gamma^+(n_t \oplus \sigma_t) \geqslant \Gamma^0(n_t \oplus \sigma_t) + (\rho_t - s)\left((s + c\rho_t)\Xi(n_t) - \int_X \nu(\mathrm{d}x) \log M_\mu(x)\right) \geqslant 0.$$

Finally, we construct the Lyapounov functional Λ^- on K^- in the following proposition whose proof, being similar to proposition 6A, we omit.

Proposition 6B. The Lyapounov functional on K^-. Define on K^- the functional

$$\Lambda^-(n \oplus \sigma) = \Lambda^0(n \oplus \sigma) + (\rho - s) \int_X \nu(\mathrm{d}x) \log M_\mu(x). \qquad (4.22)$$

Then

(a) for all $n \oplus \sigma \in K^-$, $\Lambda^-(n \oplus \sigma) \geqslant \Lambda^-(m_\mu \oplus \hat{\sigma}) = 0$ with equality iff $n \oplus \sigma = m_\mu \oplus \hat{\sigma}$;

(b) for any initial condition in K^-

$$\Lambda^-(n_t \oplus \sigma_t) - \Lambda^-(n_\delta \oplus \sigma_\delta) = -\int_\delta^t \mathrm{d}\tau\, \Gamma^-(n_\tau \oplus \sigma_\tau)$$

for any $\delta \in (0, t]$, the functional Γ^- defined as

$$\Gamma^-(n_t \oplus \sigma_t) = \Gamma^0(n_t \oplus \sigma_t) + (s - \rho_t)\Xi(n_t) \qquad (4.23)$$

(c) $\Gamma^+(n_t \oplus \sigma_t) > 0$.

4.2. *Lyapounov functional: continuity*

We now want to relate the monotonicity of Λ^0, Λ^+ and Λ^- on K^0, K^+ and K^- to the convergence of $n_t \oplus \sigma_t$ towards $m_\mu \oplus \hat{\sigma}$. In this we will make considerable use of results obtained in [6] for a dynamical model of the free Bose gas with energy cutoff. The main point is that on K^0 (i.e. when $\rho = s$) a little manipulation shows that $\mathrm{d}n_t(x)/\mathrm{d}t$ in (4.15) has the same relation to the Lyapounov functional (4.2) as does the generator (96) to the Lyapounov functional (95) in [6]. This correspondence allows us to conclude immediately that $m_\mu \oplus \hat{\sigma}$ is the terminal state for initial conditions in K^0. In view of the fact that $\rho_t \to s$, the extension of this result to initial conditions throughout $\mathscr{B}^+$ is not surprising. We will state the main result, then proceed via a number of lemmas.

First of all define

$$D := (0, \bar{x} - \underline{x}] \subset \boldsymbol{R}^+$$

$$X_\delta := [\underline{x} + \delta, \bar{x}] \subset X \qquad \forall \delta \in D$$

and the functional Λ on B^+ by

$$\Lambda(n\oplus\sigma)=\begin{cases}\Lambda^+(n\oplus\sigma) & n\oplus\sigma\in K^+\\ \Lambda^0(n\oplus\sigma) & n\oplus\sigma\in K^0\\ \Lambda^-(n\oplus\sigma) & n\oplus\sigma\in K^-.\end{cases}$$

Theorem 9. (a) If $s\geqslant s_c$ then $\forall\delta\in D$

$$\lim_{t\to\infty}\int_{X_\delta}\nu(\mathrm{d}x)|n_t(x)-m_\mu(x)|=0.$$

(b) If $s<s_c$ then (a) holds with X replacing X_δ.

Lemma 10.

$$\lim_{t\to\infty}\int_{x\in X:n_t(x)\leqslant m_\mu(x)}\nu(\mathrm{d}x)(m_\mu(x)-n_t(x))=0. \tag{4.24}$$

Proof. Let

$$I_t(\varepsilon,\delta):=\int_{x\in X_\delta:n_t(x)\leqslant m_\mu(x)(1-\varepsilon)}\nu(\mathrm{d}x)(m_\mu(x)-n_t(x)) \tag{4.25}$$

so that the integral in (4.24) is just $I_t(0,0)$. Note that $I_t(0,0)=I_t(\varepsilon,\delta)+\tilde I_t^{(1)}(\varepsilon,\delta)+\tilde I_t^{(2)}(\delta)$ where

$$\tilde I_t^{(1)}(\varepsilon,\delta)=\int_{x\in X_\delta:m_\mu(x)(1-\varepsilon)<n_t(x)\leqslant m_\mu(x)}\nu(\mathrm{d}x)(m_\mu(x)-n_t(x)) \tag{4.26a}$$

$$\tilde I_t^{(2)}(\delta)=\int_{x\in X\setminus X_\delta:n_t(x)\leqslant m_\mu(x)}\nu(\mathrm{d}x)(m_\mu(x)-n_t(x)). \tag{4.26b}$$

Now $\forall t\geqslant 0$

$$0\leqslant\tilde I_t^{(1)}(\varepsilon,\delta)\leqslant\varepsilon\int_X\nu(\mathrm{d}x)m_\mu(x) \tag{4.27a}$$

$$0\leqslant\tilde I_t^{(2)}(\delta)\leqslant\int_0^\delta\nu(\mathrm{d}x)m_\mu(x) \tag{4.27b}$$

so a glance at (2.9) tells us that

$$\lim_{\varepsilon\to0}\lim_{\delta\to0}\tilde I_t^{(1)}(\varepsilon,\delta)=\lim_{\delta\to0}\tilde I_t^{(2)}(\delta)=0 \tag{4.28}$$

uniformly in t. So it is sufficient to prove that

$$\lim_{t\to\infty}I_t(\varepsilon,\delta)=0\qquad\forall\varepsilon,\delta. \tag{4.29}$$

Making an estimate very similar to that in [6, lemma 3], we find that $\exists C_{\varepsilon,\delta}>0$ such that

$$\Gamma^0(n_t\oplus\sigma_t)\geqslant C_{\varepsilon,\delta}(I_t(\varepsilon,\delta)+(\rho_t-s))I_t(\varepsilon,\delta). \tag{4.30}$$

Furthermore, it can be shown by applying Gronwall's lemma to (3.2b) that if $I_t(\varepsilon,\delta)=d>0$, then $\exists\delta_1>0$ such that

$$I_\tau(\tfrac{1}{2}\varepsilon,\delta)\geqslant\tfrac{1}{2}d\qquad\forall\tau\in[t,t+\delta_1]. \tag{4.31}$$

Now, if (4.29) is not satisfied, then $\forall \varepsilon\delta > 0$, $\exists d > 0$ and a divergent sequence of times $\{t_k\}$ such that

$$I_{t_k}(\varepsilon, \delta) > d.$$

Choose the $\{t_k\}$ such that $t_{k+1} - t_k > \delta_1$, and t_1 such that $|\rho_{t_1} - s| \leq \frac{1}{2}d$. Then by propositions 6, 6A and 6B and because $\Gamma^{\pm} \geq \Gamma^0 \geq 0$,

$$\begin{aligned}\Lambda(n_{t_k+\delta_1} \oplus \sigma_{t_k+\delta_1}) - \Lambda(n_{t_1} \oplus \sigma_{t_1}) &= -\int_{t_1}^{t_k+\delta_1} \mathrm{d}t\, \Gamma^{\pm}(n_t \oplus \sigma_t) \\ &\leq -\int_{t_1}^{t_k+\delta_1} \mathrm{d}t\, \Gamma^0(n_t \oplus \sigma_t) \\ &\leq -\sum_{i=1}^{k} \int_{t_i}^{t_i+\delta_1} \mathrm{d}t\, \Gamma^0(n_t \oplus \sigma_t) \\ &\leq -\tfrac{1}{8} k C_{\varepsilon,\delta} d^2 \delta_1 .\end{aligned} \tag{4.32}$$

But $\Lambda(n_t \oplus \sigma_t)$ is bounded below by zero, which contradicts (4.32).

Lemma 11. For all $\delta \in D$ define

$$J_t(\delta) := \int_{x \in X_\delta : n_t(x) \geq m_\mu(x)} \nu(\mathrm{d}x)(n_t(x) - m_\mu(x))$$

Then $\forall \delta, \varepsilon_1$ and $t_0 > 0$, $\exists \tau > t_0$ such that $J_\tau < \varepsilon_1$.

Proof. It is enough to prove that $\forall \delta, \varepsilon, \varepsilon_1$, $t_0 > 0$, $\exists \tau > t_0$ such that

$$J_\tau(\varepsilon, \delta) := \int_{x \in X_\delta : n_\tau(x) \geq m_\mu(x)(1+\varepsilon)} \nu(\mathrm{d}x)(n_\tau(x) - m_\mu(x)) < \varepsilon_1 . \tag{4.33}$$

If $\sigma_0 > 0$ then by lemma 3 $\sigma_t > 0\ \forall t > 0$. Moreover, lemma 10 tells us that $\forall \varepsilon_2 > 0 \exists t_1 > t_0$ such that $\forall t > t_1$, $I_t(0,0) < \varepsilon_2$. Now suppose (4.33) is false. Then there would exist ε, δ, $d > 0$ such that $J_t(\varepsilon, \delta) > d\ \forall t > 0$. Choosing ε_2 small enough, one obtains from (3.2a) that

$$\forall t > t_1 : \frac{\mathrm{d}}{\mathrm{d}t}\sigma_t \geq k\sigma_t \qquad \text{with } k > 0$$

and hence that $\lim_{t\to\infty} \sigma_t = \infty$: a contradiction. If $\sigma_0 = 0$, one can use a similar argument by considering $\int_0^{\delta_1} \nu(\mathrm{d}x) n_t(x)$ for sufficiently small δ_1 [6].

The previous two lemmas are concerned with the particulars of the evolution. The following two lemmas concentrate on the continuity of the Lyapounov functional.

Lemma 12. Let $\{n^i \oplus \sigma^i\}^{i \in N}$ be a sequence in $\mathscr{B}^+$. Then either of the following conditions imply that $\Lambda^0(n^i \oplus \sigma^i) \to \Lambda^0(m_\mu \oplus \hat{\sigma})$:

(a) if $s \geq s_c$ then $\forall \delta \in D$:

$$\lim_{i\to\infty} \int_{X_\delta} \nu(\mathrm{d}x)|n^i(x) - m_\mu(x)| = 0 \tag{4.34}$$

(b) if $s < s_c$ then (4.34) holds with $\delta = 0$.

Proof. Follows immediately from [6, lemma 4].

Corollary 12A. In lemma 12, let $n^i \oplus \sigma^i = n_{t_i} \oplus \sigma_{t_i}$: the evolutes of some $n_0 \oplus \sigma_0$ for a divergent sequence of times $\{t_i\}$. If the conditions (a) or (b) apply, then

$$\Lambda(n_{t_i} \oplus \sigma_{t_i}) \to \Lambda(m_\mu \oplus \hat{\sigma}) \tag{4.35}$$

and hence

$$\Lambda(n_t \oplus \sigma_t) \to \Lambda(m_\mu \oplus \hat{\sigma}). \tag{4.36}$$

Proof. Examination of (4.20) and (4.22) shows that

$$\Lambda^{\pm}(n_t \oplus \sigma_t) = \Lambda^0(n_t \oplus \sigma_t) + |\rho_t - s| G_t^{\pm}(n_0 \oplus \sigma_0) \tag{4.37}$$

where for each initial condition $n_0 \oplus \sigma_0$, $G_t^{\pm}(n_0 \oplus \sigma_0)$ is positive and for each $\delta > 0$ uniformly bounded for $t > \delta$. Since $\rho_t \to s$, (4.35) follows. Finally, since $\Lambda(n_t \oplus \sigma_t)$ is monotonic in t, we have (4.36).

A converse result can be stated.

Lemma 13. If $\Lambda(n_t \oplus \sigma_t) \to \Lambda(m_\mu \oplus \hat{\sigma})$ as $t \to \infty$ then either

(a) if $s \geq s_c$ then $\forall \delta \in D$:

$$\lim_{t\to\infty} \int_{X_\delta} \nu(\mathrm{d}x) |n_t(x) - m_\mu(x)| = 0 \tag{4.38}$$

or

(b) if $s < s_c$ then (4.38) holds with $\delta = 0$.

Proof. By (4.37) it is sufficient that $\Lambda^0(n_t \oplus \sigma_t) \to \Lambda^0(m_\mu \oplus \hat{\sigma})$. But the proof for this case follows simply from [6, lemma 5].

Proof of theorem 9. Combining lemmas 10 and 11 we see that the conditions (a) or (b) of lemma 12 hold (depending on whether $s \geq s_c$ or $s < s_c$) for some sequence of evolutes $n_{t_i} \oplus \sigma_{t_i}$. Thus, by application of lemmas 12 and 13, the theorem is proved.

5. Linearised evolution

Global stability established, we can now make a more detailed examination of the return to stationarity. Again, we adapt from a treatment of stability in the free Bose gas [6]. We work again in $L^1(X, \mathrm{d}\nu)$ and set

$$n_t(x) = h_t(x) + m_\mu(x) \qquad \delta_t = \rho_t - s. \tag{5.1}$$

Then retaining terms linear in h_t and δ_t, equation (4.15) is approximated by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} h_t(x) = {}& -\delta_t + (1+\gamma e(\mu)) \int_X \nu(\mathrm{d}y) \left(\frac{h_t(y) m_\mu(x)}{1+m_\mu(y)} - \frac{h_t(x) m_\mu(y)}{1+m_\mu(x)} \right) \\ & + (1+\gamma e(\mu)) m_\mu(x)(1 - M_\mu(\underline{x})) \left(\delta_t - \int_X \nu(\mathrm{d}y) h_t(y) \right) \\ & + (1+\gamma e(\mu)) \hat{\sigma} h_t(x)(M_\mu(x) - M_\mu(\underline{x}))\end{aligned} \tag{5.2}$$

$\mathrm{d}\delta_t/\mathrm{d}t = -\delta_t$.

We now state the main theorem on linear stability. We treat only the case $s < s_c$, i.e. $\hat{\sigma} = 0$.

Theorem 14. The solution of (5.2) obeys

$$\sigma_t = \sigma_0 \exp(-\lambda_\mu t) \tag{5.3a}$$

where λ_μ, the relaxation constant, is

$$\lambda_\mu = \frac{s(1+\gamma e(\mu))}{1+m_\mu(\underset{\sim}{x})} \tag{5.3b}$$

and, furthermore,

$$\|h_t\| \leqslant F(h_0, \sigma_0) \exp(t \sup\{1, \bar{\lambda}_\mu\}) \tag{5.3c}$$

with

$$\bar{\lambda}_\mu = \frac{m_\mu(\bar{x})(1+\gamma e(\mu))}{1+m_\mu(\underset{\sim}{x})} \tag{5.3d}$$

and F depends only on the initial conditions.

Proof. First consider relaxation of the condensate. From (5.1), $\sigma_t = \delta_t - \int_X \nu(\mathrm{d}y) h_t(y)$. Combining the equations (5.2) we find that

$$\mathrm{d}\sigma_t/\mathrm{d}t = -\sigma_t(1+\gamma e(\mu))s(1-M_\mu(\underset{\sim}{x}))$$

whence (5.3a).

Now consider relaxation of h_t. Consider the subspace

$$\underset{\sim}{\mathrm{L}} = \left\{ h \in \mathrm{L}^1(X, \mathrm{d}\nu) \colon \int_X \nu(\mathrm{d}x) h(x) = 0 \right\}$$

and the operator

$$(Uh)(x) = (1+\gamma e(\mu)) \int_X \nu(\mathrm{d}y) \left(\frac{h_t(y) m_\mu(x)}{1+m_\mu(y)} - \frac{h_t(x) m_\mu(y)}{1+m_\mu(x)} \right)$$

acting in L. It is readily demonstrated that

$$\tilde{h}(x) = \|h\|_{\mathrm{L}^1} \operatorname{sgn}(h(x)) \in \mathrm{L}^\infty(X, \mathrm{d}\nu)$$

is a normalised tangent functional [15] for $h \in \underset{\sim}{\mathrm{L}}$, and that

$$\langle \tilde{h}, (U+\bar{\lambda}_\mu)h \rangle \leqslant 0. \tag{5.4}$$

Hence, by [15, theorem X48], $U+\bar{\lambda}_\mu$ generates a contraction semigroup on $\underset{\sim}{\mathrm{L}}$. Thus, for all $h \in \underset{\sim}{\mathrm{L}}$,

$$\|\exp(U_t)h\| \leqslant \exp(-\bar{\lambda}_\mu t)\|h\|. \tag{5.5}$$

Similarly, one can show that U alone generates a contraction semigroup on the whole of $\mathrm{L}^1(X, \mathrm{d}\nu)$.

Next, write (5.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t(x) = Uh_t + \vartheta\sigma_t - \mathbf{1}\delta_t \tag{5.6a}$$

with

$$\vartheta(x) = \frac{(1+\gamma e(\mu)) m_\mu(x)}{1+m_\mu(\underset{\sim}{x})}. \tag{5.6b}$$

Now since $\delta_t = \delta_0 \exp(-t)$ and $\sigma_t = \sigma_0 \exp(-\lambda_\mu t)$ we can write

$$h_t = \exp(tU)h_0 + \int_0^t \mathrm{d}r \exp[(t-r)U](\sigma_0 \exp(-r\lambda_\mu)\vartheta - \delta_0 \exp(-r)\mathbf{1}). \tag{5.7}$$

Let

$$\psi(x) = \frac{m_\mu(x)(1+m_\mu(x))}{\|m_\mu(1+m_\mu)\|}$$

where ψ is a normalised eigenfunction of U with eigenvalue 0. Thus (5.7) can be rewritten as

$$h_t = \exp(tU)[h_0 + (\sigma_0 - \delta_0)\psi] - [\sigma_0 \exp(-\lambda_\mu t) - \delta_0 \exp(-t)]\psi$$
$$+ \int_0^t \mathrm{d}r \exp[(t-r)U][\sigma_0 \exp(-r\lambda_\mu)(\vartheta - \lambda_\mu \psi) - \exp(-r)\delta_0(\mathbf{1}-\psi)]. \tag{5.8}$$

Since $h_0 + (\sigma_0 - \delta_0)\psi$, $\vartheta - \lambda_\mu \psi$ and $\mathbf{1} - \psi$ are all in $\underline{\mathrm{L}}$, then from (5.5) we have the bound

$$\|h_t\| \leqslant \exp(-\bar{\lambda}_\mu t)\|h_0 + (\sigma_0 - \delta_0)\psi\| + \sigma_0 \exp(-\lambda_\mu t) + \delta_0 \exp(-t)$$
$$+ \sigma_0 \frac{\exp(-\bar{\lambda}_\mu t) - \exp(-\lambda_\mu t)}{\lambda_\mu - \bar{\lambda}_\mu} \|\vartheta - \psi\lambda_\mu\|$$
$$+ \delta_0 \frac{\exp(-\bar{\lambda}_\mu t) - \exp(-t)}{1 - \bar{\lambda}_\mu} \|\mathbf{1} - \psi\|. \tag{5.9}$$

Since $\bar{\lambda}_\mu < \lambda_\mu$ we have (5.3*c*), unless $\bar{\lambda}_\mu = 1$, in which case the bound is of the form $Ft \exp(-t)$.

By examination of (5.3*b*) and (2.4) we see that the relaxation time for the condensate blows up as s approaches s_c from below (i.e. as $\mu \nearrow \underline{x}$). Thus the disordered phase becomes unstable at the phase transition.

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